Approaching Mean-Variance Efficiency for Large Portfolios

Yingying Li

Department of ISOM & Department of Finance

Hong Kong University of Science and Technology

Based on Joint Work with Mengmeng Ao and Xinghua Zheng

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Betterment Investment Methodology

The Betterment Portfolio Strateg

betterment.com/resources/betterment-portfolio-strategy/#diversification

II. Achieving Global Diversification with a Nobel Prize-Winning Approach to Asset Allocation

An optimal asset allocation is one that lies on the efficient frontier, which is a set of portfolios that can achieve the maximum objective for the lowest amount of risk. The objective of most long-term portfolio strategies is to maximize return, while the associated risk is measured in terms of volatility—the dispersion of those returns. In line with our investment philosophy of making systematic decisions backed by research, Betterment's asset allocation is based on a theory by economist Harry Markowitz called Modern Portfolio Theory, as well as subsequent advancements based on that theory.[1]

Introduced in 1952, Markowitz' work was awarded the Nobel Prize in 1990 after his theoretical framework and mathematical modeling informed decades of improvements in portfolio strategy construction. While there remains enormous debate (and entire sectors of financial services) devoted to portfolio construction and optimization, many

Wealthfront Investment Methodology

Wealthfront Investment Methodo ×

research.wealthfront.com/whitepapers/investment-methodology/

Allocating Assets

Wealthfront determines the optimal mix of our chosen asset classes by using Mean-Variance Optimization (Markowitz, 1952), the foundation of Modern Portfolio Theory. The output of the optimization is a collection of portfolios that generate the maximum return at each level of targeted risk, or equivalently, minimize the level of risk for a specific expected return. Collectively these portfolios form the (mean-variance) efficient frontier.

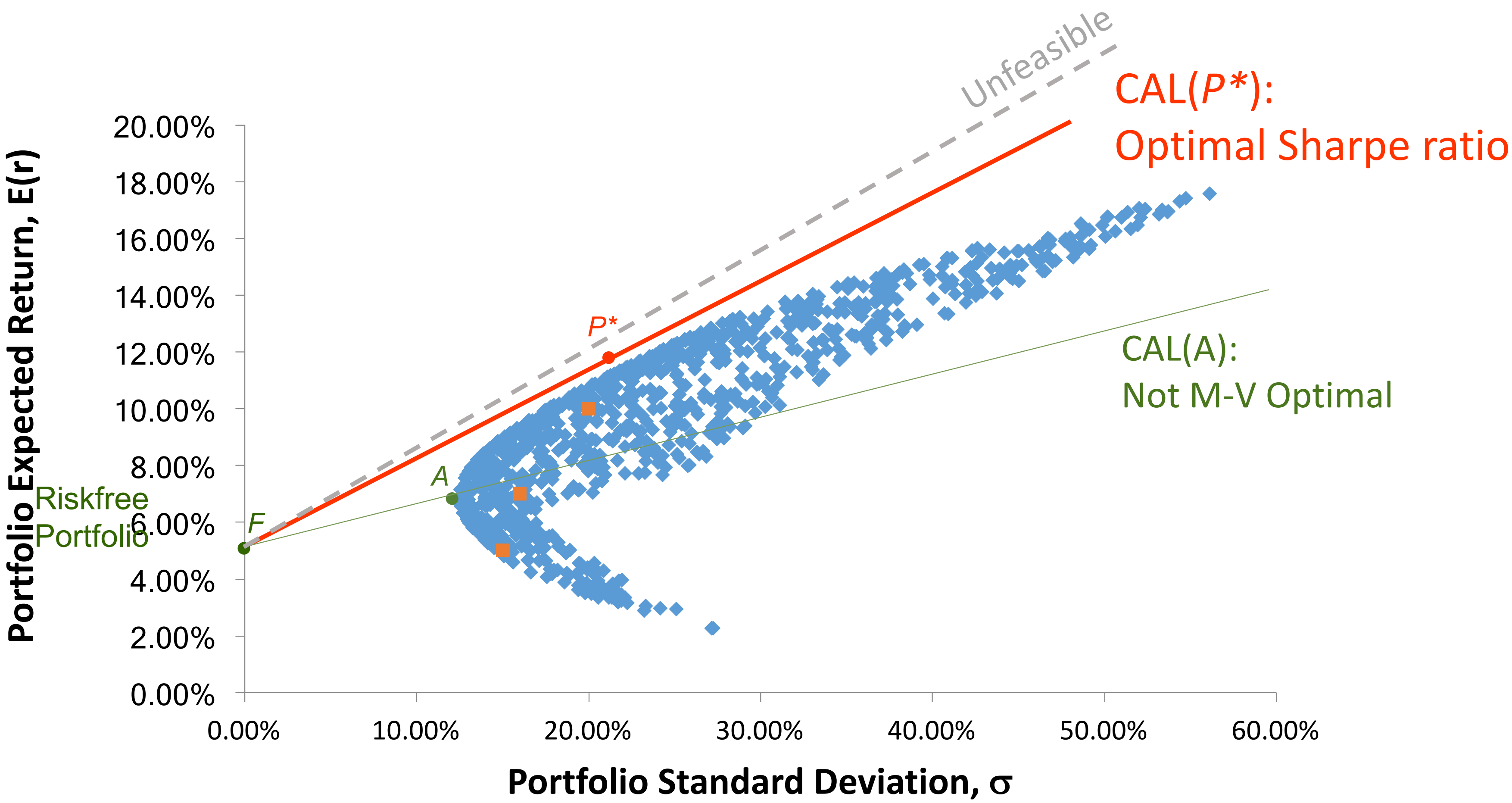
Mean-Variance Optimization

The expected return of the portfolio is a weighted average of the expected returns of the individual asset classes, μ, with the weights given by the portfolio allocations, w. The variance of the portfolio depends on the variances of the individual assets classes, but also on how they comove with one another, collectively captured by the asset class covariance matrix, Σ. To identify mean-variance efficient portfolios we solve the following quadratic programming problem:

SECTIONS

Introduction
Finding Asset Classes
Selecting Investment Vehicle
Allocating Assets
Mean-Variance Optimization
Capital Market Assumptions
Expected Returns
Variance-Covariance Matrix
Portfolio Construction
Taxable and Retirement Acc
The Benefit of Adding Risk P
Handling Small Accounts

Harry Markowiz, NOBEL PRIZE 1990


Unfeasible
CAL(*P**):
Optimal Sharpe ratio
*P**
CAL(A):
Not M-V Optimal
A
Riskfree
Portfolio
F
Portfolio Expected Return, E(r)
20.00%
18.00%
16.00%
14.00%
12.00%
10.00%
8.00%
6.00%
4.00%
2.00%
0.00%
0.00%
10.00%
20.00%
30.00%
40.00%
50.00%
60.00%
Portfolio Standard Deviation, σ

William F. Sharpe, NOBEL PRIZE 1990

Markowitz Mean-Variance Optimization

- Markowitz (mean-variance) optimization:

  maximize portfolio return given risk constraint

  $\Leftrightarrow$ minimize portfolio risk given return constraint

- The solution to Markowitz optimization is *mean-variance efficient*.

Big Data Issues
&
MAXSER

Ao, M., Yingying, L., and Zheng, X. (2019), "Approaching Mean-Variance Efficiency for Large Portfolios," The Review of Financial Studies, 32, 2890–2919.

Large Portfolio Optimization

- A large number of stocks available for investing:
  - DJIA 30; S&P 500;   Nasdaq: > 3,000;
  - FTSE 100; DAX 30; CAC 40;
  - HSI: 50;   HSCI: 500; SSE: > 1,000...
- Optimizing over a large number of assets faces intrinsic challenges.

Markowitz Mean-Variance Optimization

- Markowitz (mean-variance) optimization:

maximize portfolio return given risk constraint

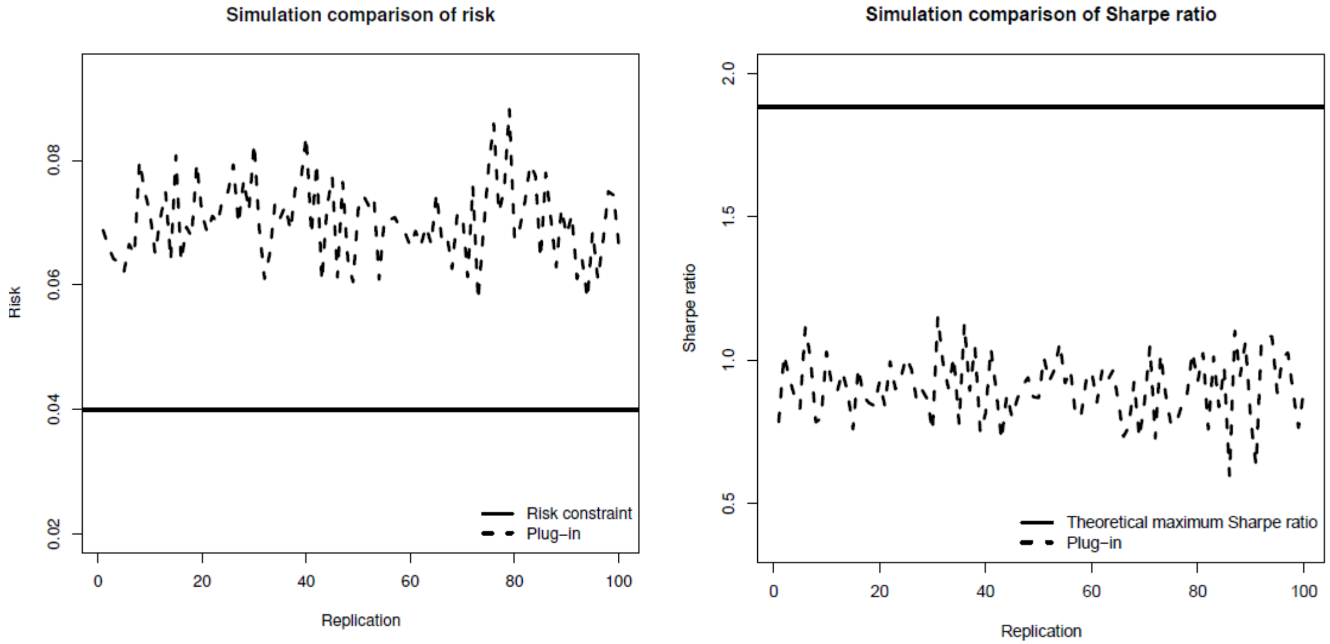
$$\boldsymbol{w}^* = \frac{\sigma}{\sqrt{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}}\Sigma^{-1}\boldsymbol{\mu} = \frac{\sigma}{\sqrt{\theta}}\Sigma^{-1}\boldsymbol{\mu}$$

where $\theta = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$ is the square of the *maximum Sharpe ratio*.

- $\boldsymbol{w}^*$ attains the maximum expected return $r^* = \sigma\sqrt{\theta}$ and Sharpe ratio $\sqrt{\theta}$.

How well does the plug-in portfolio perform?

- Recall: $$\boldsymbol{w}^* = \frac{\sigma}{\sqrt{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}}\Sigma^{-1}\boldsymbol{\mu}$$



$N = 103, T = 240$: Returns simulated from $i.i.d.$ multivariate normal with parameters calibrated from real data.

Challenges for Large Portfolios

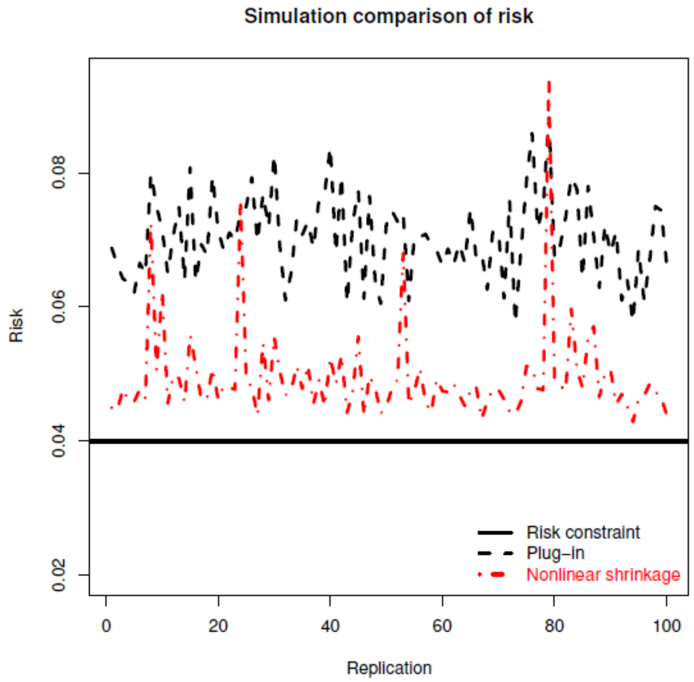
- Poor performance of the plug-in portfolio
  - "Markowitz optimization enigma": Michaud (1998)
  - Best and Grauer (1991), Chopra and Ziemba (1993), Kan and Zhou (2007) etc.
- The situation worsens as the number of assets increases
- Key reason — ***High-dimensionality*** (Ao, Li and Zheng 2019, RFS)

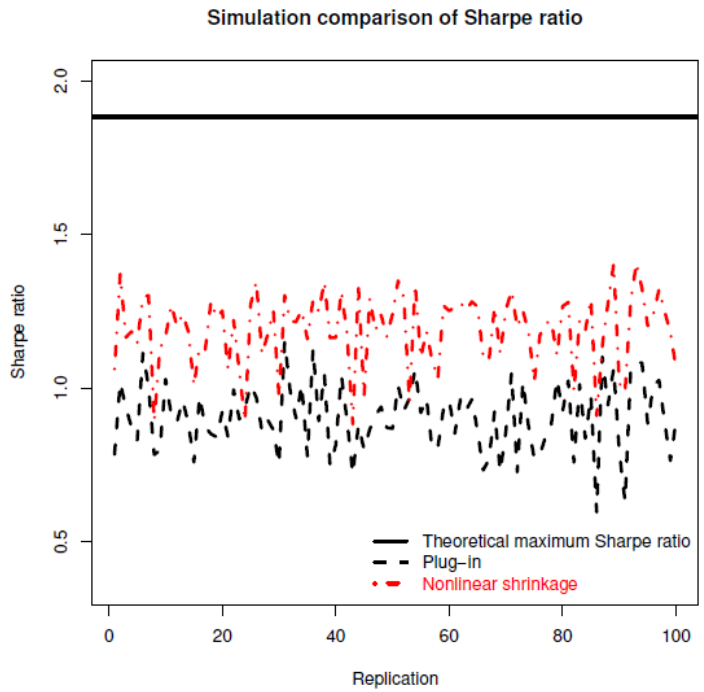
$$\frac{SR(\text{plug} - \text{in})}{SR^*} \xrightarrow{P} \sqrt{\frac{1-\eta}{1+\eta/(SR^*)^2}} < 1, \qquad as\ \frac{N}{T} \to \eta \in (0,1)$$

Alternative Methods

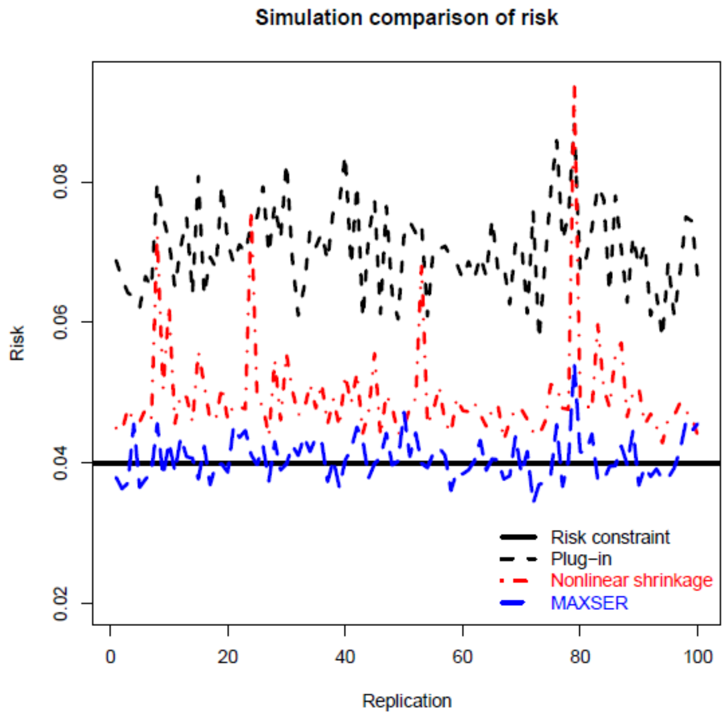
- Adjusting inputs:
  - Regularize covariance matrix or its inverse:
    - shrinkage (Ledoit and Wolf (2004, 2017); Engle et al. (2019));
    - thresholding (Bickel and Levina (2008)); CLIME (Cai et al. (2011));
    - utilizing factor structure (Fan et al. (2008, 2011, 2013); Ding et al. (2020));
    - and others...
  - Mean estimation: Black and Litterman (1991)
- Imposing constraints:
  - No-short-sale constraint (Jagannathan and Ma (2003));
  - gross-exposure/l_1 constraint (Brodie et al. (2009); Fan et al. (2012a,b));
  - 2-norm-constrained minimum variance portfolio (DeMiguel et al. (2009));
  - other non-convex constraints (Fastrich et al. (2015))

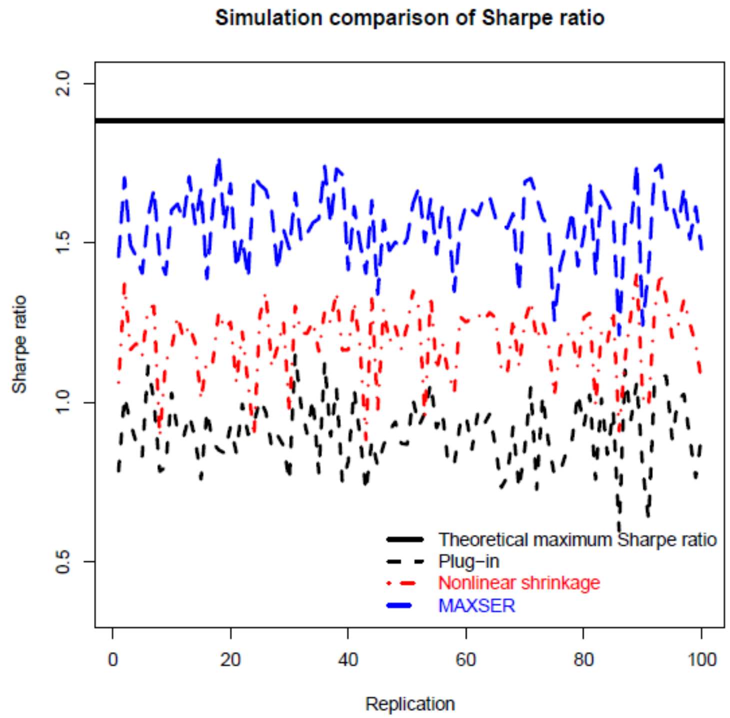
A Competitive Alternative: Nonlinear Shrinkage (Ledoit and Wolf (RFS, 2017))





Our Portfolio: MAXSER (Ao, Li and Zheng (RFS,2019)).





Key Elements of the MAXSER Approach

- Unconstrained regression representation of the mean-variance problem: $$\arg\min_{\boldsymbol{w}} E(r_c - \boldsymbol{w}'\boldsymbol{R})^2, \qquad \text{where } r_c = \frac{1+\theta}{\theta} r^* = \sigma \frac{1+\theta}{\sqrt{\theta}}$$
- Consistent estimation of response $r_c \Leftarrow \theta$
- A LASSO-type estimator of $\boldsymbol{w}^*$:

$$\widehat{w^*} = \arg\min_{\boldsymbol{w}} \sum_{t=1}^{T} (\widehat{r_c} - \boldsymbol{w}'\boldsymbol{R_t})^2 \ \textit{subject to} \ \|\boldsymbol{w}\|_1 \leq \lambda$$

- Developed for two scenarios with or without factor investing

$\boldsymbol{w}^*$ is our MAXimum - Sharpe ratio Estimated & sparse Regression (MAXSER) portfolio.

Robert Tibshirani, COPSS Presidents' award 1996

The "Nobel Prize of Statistics"

Main Result I:

Theorem 1

The MAXSER portfolio $\widehat{\boldsymbol{w}^*}$ satisfies that, as $N \to \infty$,

$$|\mu' \widehat{\boldsymbol{w}^*} - r^*| \xrightarrow{P} 0,$$

and

$$\left|\sqrt{\widehat{\boldsymbol{w}^*}' \Sigma \widehat{\boldsymbol{w}^*}} - \sigma\right| \xrightarrow{P} 0.$$

- MAXSER asymptotically *achieves the maximum expected return* and meanwhile *satisfies the risk constraint*, therefore **approaches mean-variance efficiency**!
- **First method** ever that achieves both objectives for large portfolios

Factor Models

- Capital Asset Pricing Model (CAPM)

$$r_i = \alpha_i + \beta_i\left(R_M - R_f\right) + e_i, \qquad i = 1, \dots, N.$$

- Fama-French Three Factor Model

$$r_i = \alpha_i + \beta_{i1}\left(R_M - R_f\right) + \beta_{i2} SMB + \beta_{i3} HML + e_i, \qquad i = 1, \dots, N.$$

Eugene Fama, NOBEL PRIZE 2013

With Factor Investing --
A Factor-Idiosyncratic Component Separation

- Consider the following model of returns:

$$r_i = \alpha_i + \sum_{j=1}^{K} \beta_{ij} f_j + e_i \coloneqq \sum_{j=1}^{K} \beta_{ij} f_J + u_i \,, \qquad i = 1, \ldots., N,$$

- We will invest in the N assets and the K factors
- The optimal portfolio $\boldsymbol{w}_{all}$:

$$\boldsymbol{w}_{all} \coloneqq (\boldsymbol{w}_f, \boldsymbol{w}) = \left(\sigma \sqrt{\frac{\theta_f}{\theta_{all}}} \boldsymbol{w}_f^* - \sigma \sqrt{\frac{\theta_u}{\theta_{all}}} \beta' \boldsymbol{w}_u^*, \sigma \sqrt{\frac{\theta_u}{\theta_{all}}} \boldsymbol{w}_u^* \right).$$

The MAXSER Portfolio

- Plug-in estimator of $\boldsymbol{w}_f^*$: $\widehat{\boldsymbol{w}_f^*} := \frac{1}{\sqrt{\hat{\theta}_f}} \hat{\Sigma}_f^{-1} \hat{\mu}_f$
- Estimator of $\boldsymbol{w}_u^*$:

$$\widehat{\boldsymbol{w}_u^*} = \arg\min_{\boldsymbol{w}} \frac{1}{T} \sum_{t=1}^{T} \left(\widehat{r_c} - \boldsymbol{w}' \widehat{\boldsymbol{U_t}}\right)^2 \quad subject\ to\ \|\boldsymbol{w}\|_1 \leq \lambda$$

- Final estimator of the optimal portfolio $\boldsymbol{w}_{all}$:

$$\widehat{\boldsymbol{w}_{all}} := \left(\widehat{\boldsymbol{w}_f}, \widehat{\boldsymbol{w}}\right) = \left(\sigma \sqrt{\frac{\hat{\theta}_f}{\hat{\theta}_{all}}} \widehat{\boldsymbol{w}_f^*} - \sigma \sqrt{\frac{\hat{\theta}_u}{\hat{\theta}_{all}}} \widehat{\beta}' \widehat{\boldsymbol{w}_u^*}, \sigma \sqrt{\frac{\hat{\theta}_u}{\hat{\theta}_{all}}} \widehat{\boldsymbol{w}_u^*}\right).$$

Main Result II: MAXSER with Factor Investing

Theorem 2

Under normality assumption on returns and a mild sparsity assumption on $\boldsymbol{w}_u^*$, as $N \to \infty$, the MAXSER portfolio $\widehat{\boldsymbol{w}_{all}}$ satisfies

$$\left|\widehat{\boldsymbol{w}_{all}}'\mu_{all} - r^*\right| \xrightarrow{P} 0, \text{ and } \left|\widehat{\boldsymbol{w}_{all}}'\Sigma_{all}\widehat{\boldsymbol{w}_{all}} - \sigma^2\right| \xrightarrow{P} 0,$$

Where $r^* = \boldsymbol{w}_{all}'\mu_{all}$ is the maximum expected return at risk level σ.

- MAXSER asymptotically *achieves the* ***maximum expected return*** and meanwhile ***satisfies the risk constraint***, therefore **approaches mean-variance efficiency**!

Empirical studies: Data & Rolling-window Scheme

- Data: Two asset universes
  - DJIA 30 constituents and Fama-French three factors
  - S&P 500 constituents and Fama-French three factors
- Rolling-window scheme
  - monthly rolling and rebalancing
  - risk constraint fixed to be the standard deviation of the index during the first training period
- Stock pool determination
  - DJIA 30: all constituents at each time of portfolio construction, updated monthly
  - S&P 500: yearly updated stock pools consisting of 100 randomly picked constituents

Benchmark Portfolios

Portfolio	Abbreviation
Index	Index
Equally weighted portfolio	Equally weighted
Plug-in MV on factors	Factor
Three-fund portfolio (Kan and Zhou (2007))	KZ
MV with sample cov	MV-P
MV with linear shrinkage cov (Ledoit and Wolf (2004b))	MV-LS
MV with nonlinear shrinkage cov (Ledoit and Wolf (2007))	MV-NLS
MV with nonlinear shrinkage cov adjusted for factor models (Ledoit and Wolf (2007))	MV-NLSF
GMV with linear shrinkage cov	GMV-LS
GMV with nonlinear shrinkage cov	GMV-NLS

MV = mean-variance portfolio GMV= global minimum variance portfolio

DJIA Constituents & FF3

DJIA 30 Constituents & FF3 (*Without Transaction Costs*) $T = 60$ $\sigma = 0.05$				
Period	1977-2016		1997-2016	
Portfolio	Risk	Sharpe Ratio	Risk	Sharpe Ratio
Index	0.043	0.270	0.043	0.310
Equally weighted	0.042	0.328	0.044	0.307
Factor	0.055	0.427	0.058	0.254
KZ	0.104	0.250	0.097	0.265
MV-P	0.116	0.196	0.132	0.292
MV-LS	0.070	0.132	0.077	0.376
MV-NLS	0.068	0.166	0.073	0.352
MV-NLSF	0.067	0.232	0.070	0.290
GMV-LS	0.016	0.453	0.018	0.307
GMV-NLS	0.016	0.364	0.018	0.274
MAXSER	0.060	0.556	0.064	0.567

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S&P 500 Constituents & FF3

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Period	1977-2016		1997-2016	
Portfolio	Risk	Sharpe Ratio	Risk	Sharpe Ratio
Index	0.043	0.279	0.044	0.302
Equally weighted	0.047	0.332	0.049	0.344
Factor	0.040	0.517	0.045	0.409
KZ	0.081	0.369	0.087	0.331
MV-P	0.347	0.383	0.367	0.257
MV-LS	0.079	0.248	0.078	0.093
MV-NLS	0.061	0.232	0.064	0.091
MV-NLSF	0.054	0.348	0.057	0.141
GMV-LS	0.022	0.277	0.025	0.436
GMV-NLS	0.025	0.271	0.027	0.467
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GMV-NLS	0.025	0.271	0.027	0.467
MAXSER	**0.047**	**0.667**	**0.053**	**0.591**

What about transaction costs?

- The portfolio return net of transaction cost in period t, $r_{net}(t)$ is calculated by

$$r_{net}(t) = \left(1 - \sum_j c_{t,j} \left| w_j(t+1) - w_j(t+) \right| \right)(1 + r(t)) - 1$$

- $c_{t,j}$: a cost level that measures transaction cost per dollar traded for trading asset j
- $w_j(t+1)$: weight on asset j at the beginning of period t + 1
- $w_j(t+)$: weight of asset j at the end of period t
- $r(t)$: portfolio return without transaction cost in period t

- Based on Brandt et al. (2009) and Engle et al. (2012), we set $c_{t,j}$ to be time-varying, different for individual stock and factor portfolio

DJIA Constituents & FF3

DJIA 30 Constituents & FF3 (*With Transaction Costs*) $T = 60$ $\sigma = 0.05$				
Period	1977-2016		1997-2016	
Portfolio	Risk	Sharpe Ratio	Risk	Sharpe Ratio
Index	0.043	0.270	0.043	0.310
Equally weighted	0.042	0.317	0.044	0.300
Factor	0.055	0.265	0.058	0.146
KZ	0.108	-0.134	0.098	0.040
MV-P	0.117	-0.073	0.132	0.101
MV-LS	0.071	-0.014	0.077	0.299
MV-NLS	0.069	-0.077	0.073	0.213
MV-NLSF	0.067	0.045	0.070	0.187
GMV-LS	0.016	0.313	0.018	0.213
GMV-NLS	0.017	0.079	0.018	0.066
MAXSER	0.061	0.284	0.064	**0.402**

S&P 500 Constituents & FF3

S&P 500 Constituents & FF3 (*With Transaction Costs*)	$T = 120$		$\sigma = 0.04$	
Period	1977-2016		1997-2016	
Portfolio	Risk	Sharpe Ratio	Risk	Sharpe Ratio
Index	0.043	0.279	0.044	0.302
Equally weighted	0.047	0.307	0.049	0.330
Factor	0.040	0.408	0.045	0.330
KZ	0.082	0.009	0.087	0.160
MV-P	0.349	-0.185	0.367	-0.018
MV-LS	0.079	0.066	0.078	0.011
MV-NLS	0.061	0.099	0.064	0.022
MV-NLSF	0.054	0.175	0.057	0.054
GMV-LS	0.022	0.104	0.025	0.350
GMV-NLS	0.025	0.142	0.027	0.398
MAXSER	0.048	**0.445**	0.053	**0.483**

The key ideas include transforming the original Markowitz optimization into an equivalent unconstrained optimization problem and applying the LASSO regression. The method has been found to give a superior performance in extensive empirical studies. With the risk constraint set to the same level as the risk level of SP500, we build a portfolio with MAXSER based on the US stock market. Figure 4 shows the comparison between SP500 and MAXSER in the first two quarters of this year. We see that MAXSER performs much better than SP500. It achieved a substantially higher total return and incurred a much smaller drawdown. In terms of long-term performance, during the period 2000 – 2020, MAXSER and SP500 had a similar (annualized) risk of 18%. But MAXSER delivers a much higher Sharpe ratio (0.6 vs 0.2, after deducting transaction costs).

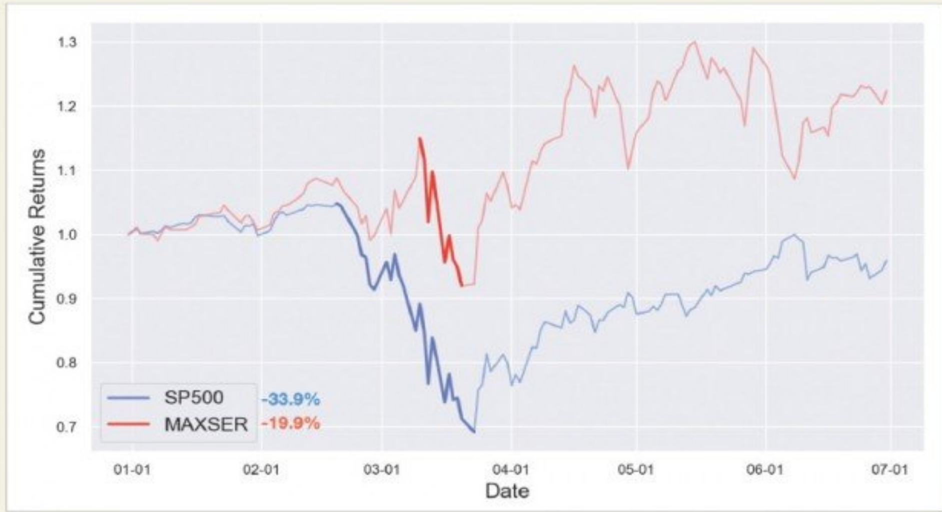
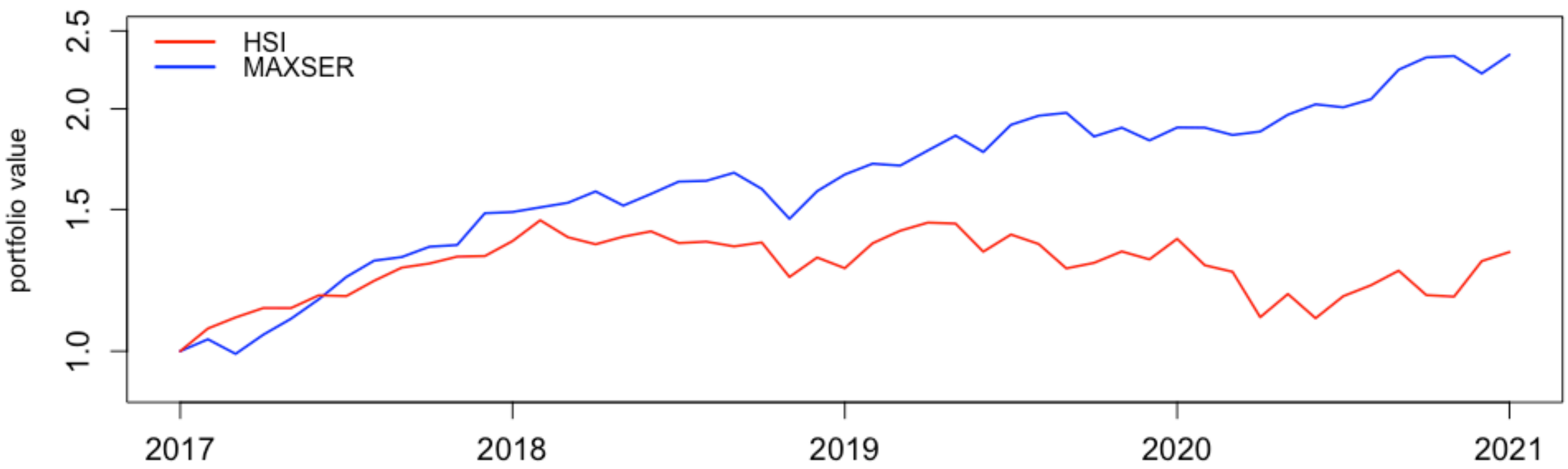


Figure 4: Comparison of SP500 and the MAXSER portfolio based on SP500 constituents in the first two quarters of 2020

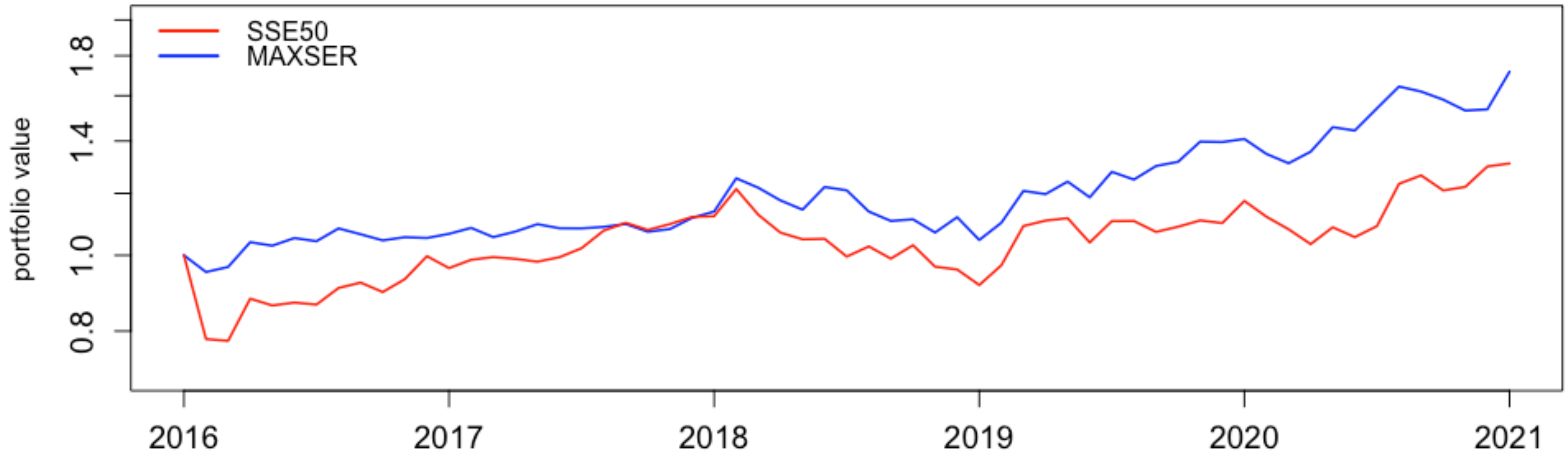
Source:
https://www.scmp.com/presented/news/hong-kong/education/topics/new-normal/article/3098737/risk-management-volatile

Portfolio value comparison: Hong Kong Stock Market



MAXSER vs HSI

Portfolio value comparison: A Share Market



MAXSER vs SSE50

Main Findings

- MAXSER asymptotically achieves the maximum Sharpe ratio and meanwhile satisfies the risk constraint
- First method ever that achieves both objectives for large portfolios
- Outstanding performance confirmed by comprehensive simulation and empirical studies

Our Lab: FinStaR@HKUST http://finstar.ust.hk

- Focus 1: High-Frequency Data
- Focus 2: High-Dimensional Financial Data
- Focus 3: Portfolio Analytics
- Focus 4: Statistical Learning, Personalization

Thank you!